In view of the complete analogy of representations (3.5), (3.6) for the mean velocity and the moments $\langle v^n\rangle$, similar expansions can be written for the latter. Assuming that the Fourier images $V^n(\eta)$ behave analytically with small wave numbers, and confining ourselves to the principal terms of the asymptotic form, we have at long times

$$\langle u\rangle \approx C_1 g(\eta, \tau),\ \langle v^2\rangle \approx C_2 g(\eta, \tau), \ldots$$
$$C_1 = \int d\eta V(\eta),\ C_2 = \int d\eta V^2(\eta), \ldots$$

For the soliton of internal waves (1.11),

$$C_1 = 4\varkappa h(1 + 1/h),\ C_2 = 8\varkappa(1 + 1/h)^2(1 - \varkappa h \operatorname{ctg} \varkappa h)$$

Using the relations along with (3.10), we can assert that, to a first approximation, we have the estimate for the velocity variance (which also remains valid for the soliton of the Benjamin-Ono equation)

$$\langle u'^2\rangle \approx \langle u_0^2\rangle + \langle v^2\rangle,\ t \to \infty$$

which shows that, at long times, the velocity pulsations are somewhat greater in the domain occupied by the soliton.

Thus, in the case of Brownian motion of the soliton, it undergoes diffusion smearing and can increase the pulsation motions of the surrounding fluid. At earlier stages, however, the effect of the presence of the soliton on the random disturbances is more considerable and more complex. For instance, immediately after switching on the random force, the disturbances increase more rapidly in front of the travelling soliton and more slowly behind it.

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THE PHENOMENA OF TURBULENT TRANSPORT AND THE RENORMALIZATION-GROUP METHOD*

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The renormalization-group (RG) method is used to study the transport of a scalar passive impurity by turbulent velocity pulses. A solution is obtained for the turbulent Prandtl numbers, which, in the case of large-scale long-term processes (the infrared limit) tends to a universal constant, which depends only on the dimensionality of the space. The version of the RG method employed enables the behaviour of the diffusion coefficient and of the Prandtl number to be found on approaching the asymptotic mode, and for it to be shown that asymptotic RG methods can be used to describe the development of turbulence in the inertial interval of the spectrum (IIS) of wave numbers.

The ideas of the RG method made their first appearance in quantum field theory /1, 2/, and have been widely used in other fields of physics. The achievements of the method are specially clear in the theory of critical effects, the laws of which are determined by the large-scale and long-term fluctuations of the order parameter. In accordance with this, the RG technique has been developed as an asymptotic approach in which the ideas about the fixed points of the RG transformation are used and the scale similitude exponents (critical indices) are found by studying the RG transformation operator, linearized near to the fixed points /3, 4/. A similar procedure has been stated, both in the context of Wilson's approach with

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successive exclusion of small-scale modes /4/, and in the context of the field-theory approach /5, 6/.

The large-scale and long-term properties of a turbulent fluid have been studied with the aid of Wilson's RG procedure /7/ or on the basis of the methods of field theory /8, 9/. Later, the results obtained by asymptotic RG methods in the infrared limit, came to be used to explain the properties of turbulence developed in the IIS, though it remained uncertain exactly how these two cases are related to each other. In particular, it has been claimed /10, 11/ that the results obtained by the RG method cannot be applied to the IIS, since this method typically makes provision for a cascade mechanism of energy transport by strong interaction between modes, whereas the use of perturbation theory to find the RG transformation operator presupposes a weak connection between the modes and does not describe a cascade process with intermode interactions, local in $\mathbf{k}$-space.

On the other hand, according to Wilson /12/, the RG method is precisely a means of describing local intermodal connections in the space of wave numbers and the cascade mechanism of interaction of modes with essentially different scales. Perturbation theory is here used only to describe a single set of mode interaction, while the cascade process is taken into account by the RG method which performs summation of a subsequence of single acts of interaction of modes of adjacent scales. The success in using the RG method to describe critical effects can be regarded as a confirmation of this point of view, though, in the case of turbulence, the question remains open as to whether the IIS belongs to the field where asymptotic methods are applicable.

In this connection, it is interesting to calculate the physical characteristics of turbulence by the RG method without using the asymptotic approach. This idea was proposed by Bogolyubov and Shirkov in quantum field theory /2/. The property of RG invariance is connected with an arbitrariness in the choice of the normalization point, while the RG method is a means of rearranging the series of formal perturbation theory and of summing an infinite subsequence of this series. With this procedure, the possibility arises in principle of finding not only the exponents of scaling similarity but also the numerical amplitude factors and obtaining the non-power dependences in the domain of incomplete similarity.

In turbulence theory also, attempts have been made to calculate the numerical amplitude factors in the context of Wilson's RG approach, based on iterative partial averaging over the small-scale modes. The method has been used to calculate the effective viscosity, which describes the average response of the velocity field to external disturbances /13/. A cut-off was then introduced into the space of wave numbers $k \leqslant \Lambda$, and the dependence of the effective viscosity on the cut-off parameter Λ was found, i.e., it was assumed that the small-scale modes with $k > \Lambda$ act on the large-scale $(k \ll \Lambda)$ like an effective viscosity $\nu^* (\mathbf{k}, \Lambda)$.

The dependence of the effective viscosity on the parameter Λ was found by solving the RG differential equation

$$\frac{\partial}{\partial \tau} \nu^* (\mathbf{k}, \Lambda (\tau)) = R (\mathbf{k}, \tau) \nu^* (\mathbf{k}, \Lambda (\tau)), \quad \Lambda (\tau) = \Lambda e^{-\tau}$$

The evolution operator $R(\mathbf{k}, \tau)$ was calculated in the lowest approximation of renormalization perturbation theory in the infrared limit $\mathbf{k} \to 0$. The result $\nu^* (0, \Lambda(\tau))$ obtained by solving this equation was identified with the effective (renormalized) viscosity at the wave number $\Lambda(\tau)$, which is inconsistent.

We proposed in /14/ a procedure for calculating the effective viscosity which makes no use of asymptotic methods. The effective viscosity $\nu_{ij}^* (\mathbf{k}, \omega)$ was defined in terms of the complete Green's function G by the relation

$$G_{ij}^{-1} (\mathbf{k}, \omega) = -i\omega\delta_{ij} + \nu_{ij}^* (\mathbf{k}, \omega) k^2 \tag{1}$$

and the static limit $\nu_{ij}^* (\mathbf{k}) = \nu_{ij}^* (\mathbf{k}, \omega)|_{\omega=0}$ was considered. Definition (1) is not identical with the commonly used definition as a means of parametrizing the influence on modes with wave number k from smaller scale modes with $k' > k$ (Heisenberg's theory /15/) or as a means of taking account in Wilson's method /4/ of modes with $k' > k$, which are excluded during the partial averaging over the small scales.

Calculation of the RG function in the lowest approximation of perturbation theory and subsequent solution of the RG compensation equation /2/ for the effective viscosity lead to the expression /14/

$$\nu_{ij}^* (\mathbf{k}) = \delta_{ij} [\nu_0^3 + {}^3/_2 A_d D_0 k^{-4}]^{1/3} \tag{2}$$

$$A_d = \frac{d-1}{8(d+2)} \frac{S_d}{(2\pi)^d}, \quad S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

(S_d is the surface area of the d-dimensional unit sphere). The correlation function of the effective random forces, used to obtain the result (2), has the form (μ is the "normalization momentum")

$$B^*(\mathbf{k}) = D_0 k^{-d} (k^2/\mu^2)^{n+\varepsilon} \tag{3}$$

Putting $n = 2$, we obtain as $\varepsilon \to 0$ the theory with logarithmic divergences, which are eliminated by viscosity renormalization (in accordance with /9/, no other renormalizations are required to eliminate divergence). The theory with logarithmic divergences corresponds to a system having the property of scale invariance (the absence of an isolated scale), which is connected with the interactions in the space of wave numbers being local and with the cascade mechanism of energy transport over the spectrum /12/.

A similar procedure can be used when considering the turbulent diffusion of a passive scalar impurity. The diffusion equation can be written as

$$\partial_t \theta + \psi_i \partial_i \theta - \varkappa_0 \Delta\theta = q \tag{4}$$

where $\theta(\mathbf{r}, t)$ is the concentration of passive impurity (or the temperature), $\varkappa_0$ is the coefficient of molecular diffusion (thermal diffusivity), q is the density of the source of passive impurity (heat), and ψ_i are the components of the turbulent fluid velocity.

We consider Green's function G_θ which describes the average linear response of the field of passive impurity to the external source

$$G_\theta(\mathbf{r}_2, t_2; \mathbf{r}_1, t_1) = \delta < \theta(\mathbf{r}_2, t_2) > / \delta q(\mathbf{r}_1, t_1) \tag{5}$$

The effective diffusion coefficient is defined in terms of the Fourier transform of the inverse Green's function

$$G_\theta^{-1}(\mathbf{k}, \omega) = -i\omega + \varkappa^*(\mathbf{k}, \omega)k^2 = -i\omega + \varkappa_0 k^2 - \Sigma(\mathbf{k}, \omega) \tag{6}$$

where $\Sigma(\mathbf{k}, \omega)$ is the correction to Green's function due to non-linear interactions, which describes the passive impurity transport by turbulent velocity pulses. Following /2/, we perform renormalization of the diffusion coefficient by means of the replacement $\varkappa_0 \to \varkappa = Z\varkappa_0$ in (6) and the addition to $\Sigma(\mathbf{k}, \omega)$ of the corresponding counter-term $(1 - Z)\varkappa_0 k^2$. The renormalization constant of the diffusion coefficient Z is found from the condition that, at the point of normalization $\mathbf{k} = \mu$, $\omega = 0$, the effective diffusion coefficient must be the same as the renormalized coefficient:

$$\varkappa^*(\mathbf{k}, \omega)|_{k=\mu,\, \omega=0} = \varkappa \tag{7}$$

i.e., the correction to the renormalized diffusion coefficient at the point of normalization must be zero.

In the lowest approximation of renormalized perturbation theory we have

$$\Sigma(\mathbf{k}, \omega) = \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{d\omega'}{2\pi} V_i(\mathbf{k}) G_\theta^{(0)}(\mathbf{q}, \omega') C_{ij}^{(0)}(\mathbf{k} - \mathbf{q}, \omega - \omega') V_j(\mathbf{q}) \tag{8}$$

$$G_\theta^{(0)}(\mathbf{k}, \omega) = [-i\omega + \varkappa k^2]^{-1}, \quad V_j = ik_j. \tag{9}$$

$$C_{ij}^{(0)}(\mathbf{k}, \omega) = P_{ij}(\mathbf{k}) B^*(\mathbf{k}) [\omega^2 + \nu^2 k^4]^{-1}$$

$$P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$$

After substituting (9) into (8) and integrating with respect to ω', we obtain, with $\omega = 0$,

$$\Sigma(\mathbf{k}, 0) = -\frac{k^2 P_{ij}(\mathbf{k})}{2\nu} \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{q_i q_j B^*(\mathbf{k} - \mathbf{q})}{(\mathbf{k} - \mathbf{q})^4} \frac{1}{\nu(\mathbf{k} - \mathbf{q})^2 + \varkappa q^2}$$

Using the form (3) for B^* and performing the integration with respect to $\mathbf{q}$ by the method of calculating the Feynman integrals /2/, we obtain

$$\Sigma(\mathbf{k}, 0) = -\frac{k^2 (d-1) D_0 \Gamma(-\varepsilon) (k^2)^{-\varepsilon}}{4 (4\pi)^{d/2} \nu (\nu + \varkappa) \mu^{4-2\varepsilon} \Gamma(d/2 - \varepsilon)} \times \int_0^1 \alpha^{d/2-1-\varepsilon} \left[\frac{\varkappa (1-\alpha)(\alpha\varkappa + \nu)}{\nu(\nu + \varkappa)} \right]^{-\varepsilon} d\alpha$$

After passing to the limit as $\varepsilon \to 0$, integrating with respect to α, and subtracting the counter-term that ensures satisfaction of the normalization condition (7), we find the expression for the effective diffusion coefficient in the lowest approximation of renormalized perturbation theory

$$\varkappa^*(\mathbf{k}, 0) = \varkappa - \frac{D_0 B_d}{\nu(\nu + \varkappa)\mu^4} \ln \frac{k^2}{\mu^2}, \quad B_d = \frac{d-1}{4d} \frac{S_d}{(2\pi)^d} \tag{10}$$

We define the effective Prandtl number by the relation

$$\mathrm{Pr}^*(\mathbf{k}) = \nu^*(\mathbf{k}, 0) / \varkappa^*(\mathbf{k}, 0) \tag{11}$$

To calculate this number in the context of renormalization perturbation theory, using arguments of dimensionality and the condition for isotropy, we have

$$[\mathrm{Pr}^* (\mathbf{k})]^{-1} = F\,(k^2/\mu^2,\ k_d^2/\mu^2,\ \varkappa/\nu),\ \ k_d D_0^{1/4} \nu_0^{-3/4} \tag{12}$$

From the normalization condition (7) we also have

$$F\,(1,\,y,\,g) = g \tag{13}$$

To find the function $F\,(x,\,y,\,g)$ we use the RG method, so that, starting from the lowest approximation of perturbation theory for $F\,(x,\,y,\,g)$, we can refine this function by summation of an infinite subsequence of the series of perturbation theory.

The condition for RG invariance, which reflects the arbitrariness in the choice of normalization point μ, can be written as /14/

$$F\,(k^2/\mu^2,\ k_d^2/\mu^2,\ \varkappa/\nu) = F\,(k^2/\mu_1^2,\ k_d^2/\mu_1^2,\ \varkappa_1/\nu_1) \tag{14}$$

Putting $k = \mu$ in (14) and using the normalization condition (13), we find that

$$\varkappa_1/\nu_1 = F\,(\mu_1^2/\mu^2,\ k_d^2/\mu^2,\ \varkappa/\nu) \tag{15}$$

Substituting (15) into (14), we obtain the RG functional equation /2/

$$F\,(x,\,y,\,g) = F\,(x/t,\ y/t,\ F\,(t,\,y,\,g)) \tag{16}$$
$$(k^2/\mu^2 = x,\ k^2/\mu^2 = y,\ \mu_1^2/\mu^2 = t,\ \varkappa/\nu = g)$$

Differentiating (16) with respect to t and then putting $t = 1$, we obtain the RG differential equation, equivalent to (16):

$$\{x\,\partial/\partial x + y\,\partial/\partial y - \beta\,(y,\,g)\partial/\partial g\}F\,(x,\,y,\,g) = 0 \tag{17}$$
$$\beta\,(y,\,g) = \partial F\,(x,\,y,\,g)/\partial x\,|_{x=1}$$

In accordance with the RG method, we find the so-called RG function $\beta\,(y,\,g)$ by starting from the lowest approximation of renormalization perturbation theory. Using (3), (10), and definition (17), we obtain

$$\beta\,(y,g) = \frac{y^2}{1 + {}^3/_2 A_d y^2}\left[A_d g - \frac{B_d}{1+g}\right]$$

Knowing $\beta\,(y,\,g)$, the solution of Eq.(17) is found by the method of characteristics and is given implicitly by the relation

$$[\sigma_- - F]^{a_-}\,[\sigma_+ + F]^{a_+}\,[1 + 3A_d y^2/(2x^2)] = C \tag{18}$$
$$\sigma_\pm = {}^1/_2\,(\sqrt{1 + 4B_d/A_d} \pm 1),\quad a_\pm = 3\sigma_\mp/(\sigma_+ + \sigma_-)$$

If we additionally require that, in the small-scale domain $(k \to \infty)$ the effective Prandtl number Pr^* transforms into the molecular number $\mathrm{Pr}_0 = \nu_0/\varkappa_0$, i.e.,

$$\lim_{x\to\infty} F\,(x,\,y,\,g) = \mathrm{Pr}_0^{-1}$$

then the constant C in (18) can be found from this supplementary condition, and Eq.(18) takes the form

$$\left[\frac{\sigma_- - F}{\sigma_- - \mathrm{Pr}_0^{-1}}\right]^{a_-}\left[\frac{\sigma_+ + F}{\sigma_+ + \mathrm{Pr}_0^{-1}}\right]^{a_+} = \left[1 + \frac{3A_d y^2}{2x^2}\right]^{-1} \tag{19}$$

Relation (19) is the implicitly written solution of the RG differential Eq.(17) for the function $F\,(x,\,y,\,g)$ defining the effective Prandtl number. In accordance with (3), (9) and (18), this number depends on the molecular Prandtl number $\nu_0/\varkappa_0$, the ratio k/k_d, and the dimensionality d of the space.

It follows from our solution that, in the corresponding **inertial** interval $k/k_d \ll 1$ of the "large-scale" domain, the turbulent Prandtl number has the universal asymptotic form

$$\lim \mathrm{Pr}^* (\mathbf{k}) = 2[\sqrt{1 + 8\,(d+2)/d} - 1]^{-1} \tag{20}$$

which is independent of the molecular Prandtl number and of the nature of the transport process (diffusion, thermal conductivity, etc.). The asymptotic result (20) was earlier obtained in the context of Wilson's approach /16/ and by means of field-theory RG /17/. The result was later reproduced in /18/, though an apparently unsatisfactory approach was then used, when the effective transport coefficients $\nu^*\,(\mathbf{k},\,\Lambda)$ and $\varkappa^*\,(\mathbf{k},\,\Lambda)$ were calculated with $\mathbf{k} = 0$, while

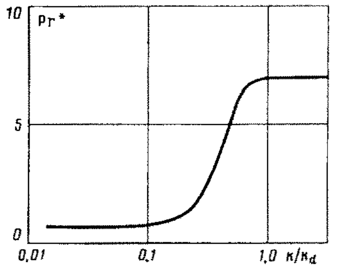

Fig.1

the boundary Λ separating the "fast" and "slow" modes was then identified with the wave number k (see also /13/).

In Fig.1 we show a curve of the effective (turbulent) Prandtl number against the wave number k with $d = 3$ for the model example $Pr_0 = 7$ (the molecular diffusion for water at $20^{o}C$). It can be seen that there is a quite rapid transition (within one decimal order) from one asymptotic state, for which $k/k_d \ll 1$ and the turbulent Prandtl number is independent of the molecular number (inertial interval), to the other for which $k/k_d \gg 1$, where the transport is realized by molecular motion (dissipation interval). We can assume from this that the infrared RG asymptotic technique can be used to study the properties of the developed turbulence in the intertial interval of the spectrum, while the numerical amplitude factors are determined by the position of the fixed points of the RG transformation, while the exponents of the power behaviour are found from the properties of the transformation, linearized close to these points.

There have recently been discussions of the connection of the RG method with renormalized perturbation theory when describing turbulence, and of the role of local and non-local intermode interactions /10, 19/. It must be said that, in these discussions, the RG method has been solely understood to be Wilson's approach, which consists in the successive reduction of the number of modes considered in the multimode system, by averaging over a narrow band of the spectrum, the small-scale and rapidly variables modes /12/, and using methods of perturbation theory to find the average influence of the excluded modes on the remainder. It is this statement of the RG method that we used in the theoretical analysis of turbulence*. (*E.V. Teodorovich, Methods of field theory in the renormalization group in statistical hydrodynamics, Preprint 302, Inst. Problem Mekhaniki, Akad. Nauk SSSR, Moscow, 1987.) As the group parameter we took the cut-off parameter in the space of wave numbers (the boundary between the fast and slow modes). A similar procedure can be used to find the exponents of the power behaviour, by using asymptotic methods of non-linear mechanics, i.e., analysing the behaviour close to a fixed point of the RG transformation /3, 4/. It was correctly pointed out by Kraichnan /10/ that this approach, while it enables the exponents of the power behaviour to be found, does not give the numerical amplitude factors. However, the power exponents can be found from simple physical considerations which use the concept of turbulent viscosity, and in this sense the use of the RG method provides nothing new. Kraichnan emphasized also that the lowest approximations of perturbation theory are unsuitable for describing cascade processes in the inertial interval, so that the RG method does not give a more rigorous basis for the exponents of the scaling similarity. It has also been claimed by others /11, 18/ that the RG method in which the lowest approximations of perturbation theory are used does not describe the energy stage.

In the statement of the RG method used in the present paper, and proposed as long ago as 1955 by Bogolyubov and Shirkov in quantum field theory /2/, the property of RG invariance is used to improve the results of perturbation theory and enables us to perform the summation of an infinite subsequence of the series of perturbation theory. In the field statement, the objects that are considered from the start are the averaged characteristics of the hydrodynamic fields, such as Green's functions and the statistical moments of differential orders, and there is no partial averaging operation. The RG invariance is linked with the arbitrariness in the renormalization procedure when constructing the renormalized theory. This lack of uniqueness has an inherent physical justification and is not connected with the presence of divergences; it reflects the fact that the physical results are independent of the method of specifying the initial and boundary conditions (functional similarity /20/).

The RG method is a means of considering multimode systems with strong interaction between the modes; according to the hypothesis of /21/, these interactions show themselves in a tendency to selfsimilarity, which is a consequence of the local nature of the interactions and the cascade mechanism of energy transport over the spectrum. As applied to such systems, the RG method in the statement of /2/ means that an individual link of the cascade process is calculated according to perturbation theory, while the cascade chain is obtained by the summation of the individual links with the aid of RG. This procedure is similar to the way in which, in the theory of continuous Lie groups, the finite transformations are constructed by using the generators of infinitesimal transformations which form the Lie algebra. In accordance with this analogy, the role of generators is played by the RG functions $\beta(y, g)$. Though the function $F(x, y, g)$ is determined in the lowest approximation of perturbation theory by the interaction of the mode with wave vector μ with all the remaining modes, i.e., account is taken of both local and non-local interactions in k-space, the derivative of this function at the normalization point (the RG function) is determined solely by the interaction with the modes of closely adjacent scales. The solution of the RG differential Eq.(17) corresponds to the summation of infinitely many individual infinitesimal links of the cascade chain.

To above field formulation of the RG method, used by the author in the present paper and in /14/, is a way of describing cascade processes mathematically. It enables the behaviour of the physical quantities to be found, not only in the asymptotic domain as $k \to 0$, but also in the wider domain of wave-number space where the cascade process is realized.

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